PERGAMON

International Journal of Solids and Structures 38 (2001) 5727–5745

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Bending of large curvature beams. II. Displacement method approach

Lando Mentrasti *

Facoltà di Ingegneria, Instituto di Scienza e Tecnica delle Costruzioni, Università degli Studi di Ancona, Via Brecce Bianche, 60131 Ancona, Italy

Received 28 December 1999

Abstract

A large curvature circular beam subject to *uniform bending* in its plane is investigated by a *displacement method*. This is the companion of a previous paper in which the same subject has been investigated by a *stress method* approach (Part I, Int. J. Solids Struct. 38 (2001) 5703–5726).

The functional form of the three-dimensional displacement field is determined exactly: the cross-section remains plane and rotates about the Z-axis, while each longitudinal fiber remains circular. This result is independent of the constitutive equation of the material, provided it is compatible with the uniform bending hypothesis.

The equilibrium equation (in several form) and the boundary conditions are derived for the linear elastic and homogeneous beam: they constitute an unstable degenerate boundary value problem with a structure similar to that found in Part I. The non-variational nature of this kind of problem is also detected.

Finally, the relationship with the potential stress function ψ is derived, governed by a *hyperbolic* second order partial differential equation (another unusual occurrence appearing in bending problem).

Keywords: Curved beam; Uniform bending; Multi-connected cross-section; Compatibility equations; Potential stress function; Degenerate boundary conditions; Exact 3D elastic solution

1. Introduction

A cylindrical curved beam, with large radius of curvature with respect to its width, loaded by bending moments at its ends is studied as a three dimensional (3D) linear problem using the *displacement method*.

This is the companion of a previous paper in which the same subject was investigated by a *stress method* approach (Mentrasti, 2001; quoted as Part I in the sequel) to which reference can be made for an introduction of the problem and for a precise definition of *uniform* thermo-mechanical state.

The principal assumption is the *uniform bending* hypothesis: the mechanical problem is assumed to be *invariant* under the group $\mathcal{G}_Z$ of rigid rotations about the Z global axis; from an analytical point of view, the

* Fax: +39-071-220-4576.

E-mail address: mentrasti@popcsi.unian.it (L. Mentrasti).

PII: S0020-7683(00)00362-0

Nomenclature

$\mathscr{A}$ cross-section domain, N-connected, with boundary $\partial\mathscr{A}$
$\mathrm{Int}(\mathscr{L})$ simply connected domain enclosed within the curve $\mathscr{L}$
$c_i \subset \partial\mathscr{A}$ ith regular curves of the boundary, $i = 1 \dots N$
b_{00} bending constant, Eq. (2.41)
E, G, v elastic constants
F auxiliary displacement potential function, Eqs. (4.28)–(4.35)
σ_{ij}, ε_{ij} stress and strain components in a cylindrical system of coordinate
$n := [n_\rho \; n_z]^{\mathrm{T}}$ components of the outward normal, $\boldsymbol{n}$, at $\partial\mathscr{A}$
$t := [-n_z \; n_\rho]^{\mathrm{T}}$ components of the tangent unit vector, $\boldsymbol{t}$, at $\partial\mathscr{A}$
$u(\rho, \varphi, z) := [u \quad v \quad w]^{\mathrm{T}}$ components of the displacement field $\boldsymbol{u}$, in cylindrical coordinates
α_{c_i} geometric constant, Eq. (4.22)
κ_i integration constant (relevant to the ith boundary)
ψ potential stress function
φ longitude of the current cross-section
Φ generalized potential displacement function
$\Lambda := \rho\partial_\rho(1/\rho)\partial_\rho + \partial_{zz}$ field operator, Eq. (4.7)
$\Lambda^* := (1/\rho)\partial_\rho\rho\partial_\rho + \partial_{zz}$ field operator, Eq. (3.13)
$\Omega := \partial_\rho(1/\rho)\partial_\rho\rho + \partial_{zz}$ field operator, Eq. (3.11)
$\{O, \rho, z\}$ system of coordinates in the plane of the cross-section
$\{O, \rho, \varphi, z\}$ cylindrical system of coordinates
$\{O, X, Y, Z\}$ global system of coordinates (Z is the axis of revolution)
$(\;)_{,\xi}$ partial derivative with respect to the variable ξ
$\square$ end of a proof

components of the *strain* tensor in cylindrical coordinates are independent of the φ variable that locates the cross-section $\mathscr{A}$ along the beam (cf. Part I, Fig. 1).

Moreover, $\gamma_{\rho\varphi}$ and $\gamma_{\varphi z}$ are assumed to be nil (bending/shear–torsion *uncoupling*).

The question here is not to duplicate previously established results; on the contrary, the investigation aims at ascertaining whether the *oblique degenerate unstable* boundary value problem (BVP) attained in that formulation can be bypassed using an alternative method or if it is a proper characteristic of the curved beam under bending. It must be clearly stated that, for this kind of problems, it is not a matter of regularity of the solution consequent to assuming regularity of data and boundary geometry. Rather, in these cases the BVP is *no longer Fredholm type* and (*infinite dimensional*) additional *solutions* or *conditions* may appear (cf. Popivanov and Pagachev, 1997).

In a first step, a functional form of the *displacement* field for the 3D problem is determined *exactly*, *without specification of the material* (e.g. elastic, plastic, viscous, non-homogeneous, etc.): the main result is that the cross-section, while deforming transversely, remains plane and rotates about the Z-axis, proportionally to φ.

The only necessary conditions on the constitutive equation are its consistency with the *basic* uniform bending hypothesis and the bending/shear–torsion uncoupling (an explicit discussion of such restrictions is not given in this paper).

The governing equilibrium equation and the boundary conditions (BCs) are then derived, leading to an *unstable degenerate* BVP governed by the same differential operator appearing in the stress approach (Part I, Section 5).

Mitchell derived one of the field equations on the basis of an a priori assumption on the displacement field. The degenerate character of BC is however *presented here for the first time*, along with a possible solution.

In conclusion, the displacements are explicitly determined in terms of an auxiliary function F, governed by a simple *hyperbolic* partial differential equation (PDE), *related to the potential stress function* ψ defined in Part I.

2. Reduction of the displacement field functions

The displacement components $[u \quad v \quad w]^{\mathrm{T}} =: \mathrm{u}(\rho, \varphi, z)$, along the axes of the cylindrical system of coordinates (ρ, η and z, respectively), are assumed as the unknowns of the problem.

The idea is to use the *strain definition* and the *compatibility* equations to determine the *dependence* of $\boldsymbol{u}$ on a *minimal number of functions* (denoted, in the sequel, with lower case Latin letter a, b, etc., in order of appearance; their dependence on the ρ, η, z variables is always explicitly shown in this paragraph).

2.1. Consequences of the compatibility equations

Strictly speaking, the compatibility equations (Part I, Appendix A, Eqs. (A.1)–(A.6)) would be identically fulfilled by every regular enough displacement field $\boldsymbol{u}(\rho, \varphi, z)$. Nevertheless, they can be used to specify the *functional dependence* of the components on the variables:

On one hand, the compatibility equations in cylindrical coordinates cannot contain any derivative with respect to φ: Eqs. (3.1)–(3.6) in Part I, named there as *I–VI*, are precisely the result achieved in such a way.

On the other hand, a generic function assumed for $\boldsymbol{u}$ does not necessarily satisfy these relationships; in fact, expanding *I–VI* as explicit functions of u, v, w, the following conditions result:

$$I \to v_{,\varphi zz} = 0, \tag{2.1}$$

$$II \equiv 0, \tag{2.2}$$

$$III \to v_{,\rho\rho\varphi} = 0, \tag{2.3}$$

$$IV \to u_{,\varphi zz} - \rho^2 \left(\frac{1}{\rho^2} w_{,\varphi z} \right)_{,\rho} = 0, \tag{2.4}$$

$$V \to -(\rho \; u_{,\varphi z})_{,\rho} + \rho^2 \left(\frac{1}{\rho} w_{,\rho,\varphi} \right)_{,\rho} = 0, \tag{2.5}$$

$$VI \to v_{,\rho\varphi z} = 0. \tag{2.6}$$

2.2. Consequences of $\varepsilon_{ij,\varphi} \equiv 0$

Since $\varepsilon_\rho := u_{,\rho}$, from $\varepsilon_{\rho,\varphi} \equiv 0$ follows that $u_{,\rho\varphi} = 0$; this PDE can be quickly integrated as

$$u(\rho, \varphi, z) = u_{\langle \rho z \rangle}(\rho, z) + u_{\langle \varphi z \rangle}(\varphi, z), \tag{2.7}$$

in which, henceforth, $\upsilon_{\langle \lambda\mu \rangle}$ denotes the part of υ depending on the variables λ and μ only.

Similarly, $\varepsilon_{z,\varphi} \equiv 0$ gives

$$w(\rho, \varphi, z) = w_{\langle \rho\varphi \rangle}(\rho, \varphi) + w_{\langle \rho z \rangle}(\rho, z), \tag{2.8}$$

in which $\varepsilon_z := w_{,\rho}$.

Since $\varepsilon_\varphi := (v_{,\varphi} + u)/\rho$, using Eq. (2.7) the condition $\varepsilon_{\varphi,\varphi} \equiv 0$ gives

$$v_{,\varphi\varphi} + u_{\langle\varphi z\rangle,\varphi} = 0, \tag{2.9}$$

from which, deriving with respect to ρ, results in

$$v_{,\rho\varphi\varphi} = 0. \tag{2.10}$$

2.3. The v component

The conditions (2.1), (2.3), (2.6), (2.10) on v permit determining its functional dependence with respect to the variables; Appendix A gives a detailed proof of the attained result:

$$v = b_{00}\rho\varphi + b_2(\rho) + b_3(\varphi) + za_{0,\varphi}(\varphi) + a_2(z) + v_{\langle\rho z\rangle}(\rho, z). \tag{2.11}$$

The shear strain $\gamma_{\rho\varphi}$ and $\gamma_{z\varphi}$ are assumed to be nil by the bending/*shear–torsion* uncoupling (cf. Part I, Section 2). In particular

$$2\varepsilon_{\rho\varphi} := u_{,\varphi}/\rho + \rho(v/\rho)_{,\rho} = 0 \tag{2.12}$$

must hold; using the results (2.7) and (2.11), this equation writes

$$\rho^2\left[\frac{1}{\rho}b_2(\rho) + \frac{1}{\rho}v_{\langle\rho z\rangle}\right]_{,\rho} + u_{\langle\varphi z\rangle,\varphi} - b_3(\varphi) - za_{0,\varphi}(\varphi) - a_2(z) = 0, \tag{2.13}$$

that, after a derivative with respect to ρ, becomes

$$\left\{\rho^2\left[\frac{1}{\rho}(b_2(\rho) + v_{\langle\rho z\rangle})\right]_{,\rho}\right\}_{,\rho} = 0. \tag{2.14}$$

This relationship is reckoned as a homogeneous ordinary differential equation (ODE), whose general integral is

$$b_2(\rho) + v_{\langle\rho z\rangle} = -\mathrm{d}(z) + \rho e(z), \tag{2.15}$$

from which $v_{\langle\rho z\rangle} = -\mathrm{d}(z) - b_2(\rho) + \rho e(z)$. Therefore, the relationship (2.11) can be finally completed as

$$v(\rho, \varphi, z) = b_{00}\rho\varphi + b_3(\varphi) + za_{0,\varphi}(\varphi) + a_3(z) + \rho e(z), \tag{2.16}$$

where $\mathrm{d}(z) + a_2(z)$ is collected in $a_3(z)$.

2.4. The w component

Consider now that $2\varepsilon_{\varphi z} := v_{,z} + w_{,\varphi}/\rho = 0$; this equation, using Eqs. (2.16) and (2.8), writes

$$a_{0,\varphi}(\varphi) + a_{3,z}(z) + \rho e_{,z}(z) + \frac{1}{\rho}w_{\langle\rho\varphi\rangle,\varphi} = 0. \tag{2.17}$$

By a derivative with respect to ρ

$$e_{,z}(z) + \left(\frac{1}{\rho}w_{\langle\rho\varphi\rangle}\right)_{,\rho\varphi} = 0 \tag{2.18}$$

is first obtained; a subsequent derivative with respect to z gives $e_{,zz}(z) = 0$, from which $e(z) = e_0 z + e_1$. Substituting for $e(z)$ in Eq. (2.18) and integrating the resulting PDE, $(w_{\langle\rho\varphi\rangle}/\rho)_{,\rho\varphi} + e_0 = 0$,

$$w_{\langle\rho\varphi\rangle} = -e_0\rho^2\varphi + h_0(\rho) + \rho h_1(\varphi) \tag{2.19}$$

is attained.

However, the original not derived Eq. (2.17) must be fulfilled by $e(z)$ and $w_{\langle\rho\varphi\rangle}$; the relevant condition is $a_{0,\varphi}(\varphi) + a_{3,z}(z) + \rho h_{1,\varphi}(\varphi) = 0$, from which it is easy to see that

$$a_3(z) = a_{31}z + a_{32}, \tag{2.20}$$

$$h_1(\varphi) = -a_0(\varphi) - a_{31}\varphi - a_{30}. \tag{2.21}$$

The required functional form for $w_{\langle\rho\varphi\rangle}$ is then achieved

$$w_{\langle\rho\varphi\rangle}(\rho,\varphi) = -e_0\rho^2\varphi + h_2(\rho) - a_{31}\rho\varphi - \rho a_0(\varphi), \tag{2.22}$$

after gathering $h_0(\rho)$ and $\rho\ a_{32}$ in $h_2(\rho)$.

2.5. The u component

The last consideration about the strain component is that $2\varepsilon_{\rho z} := w_{,\rho} + u_{,z}$ is independent of φ. Employing the expression for the u and w displacement obtained in the previous steps, the condition $\varepsilon_{\rho z,\varphi} = 0$ gives

$$-2e_0\rho - a_{31} - a_{0,\varphi}(\varphi) + u_{\langle\varphi z\rangle,\varphi z} = 0. \tag{2.23}$$

Then $e_0 = 0$ must hold; besides, $u_{\langle\varphi z\rangle}$ can be quickly determined by two quadratures as

$$u_{\langle\varphi z\rangle} = a_{31}\varphi z + z a_0(\varphi) + a_4(\varphi) + a_5(z). \tag{2.24}$$

2.5.1. Equations to be restored

Now, Eq. (2.12), $\varepsilon_{\rho\varphi} = 0$, must be reconsidered: it is not yet fulfilled since only its derivative is used hitherto (cf. Eq. (2.14)). Using the foregoing expressions for u and w, Eq. (2.12) writes

$$a_{31}z + a_{4,\varphi}(\varphi) - b_3(\varphi) - a_3(z) = 0, \tag{2.25}$$

from which it is easy to draw, in the usual manner, that

$$a_3(z) = a_{31}z + c_1, \tag{2.26}$$

$$a_4(\varphi) = c_1\varphi + \int b_3(\varphi)\,\mathrm{d}\varphi + c_2. \tag{2.27}$$

For the same reason, also the equation $\varepsilon_{\varphi,\varphi} = 0$ should be reconsidered (cf. Eq. (2.10)); however, this is postponed after the simplifications attained in the next section.

At the end of this section, it is perhaps useful to summarize the achieved results:

$$u_{\langle\varphi z\rangle} = a_{31}\varphi z + z a_0(\varphi) + a_5(z) + c_1\varphi + c_2 + \int b_3(\varphi)\,\mathrm{d}\varphi, \tag{2.28}$$

$$v(\rho,\varphi,z) = b_{00}\rho\varphi + b_3\varphi + z a_{0,\varphi}(\varphi) + c_1 + a_{31}z + e_1\rho, \tag{2.29}$$

$$w_{\langle\rho z\rangle} = h_2(\rho) - a_{31}\rho\varphi - \rho a_0(\varphi). \tag{2.30}$$

2.6. Consequences of the compatibility equations

The compatibility condition $III = 0$ is identically satisfied by the displacement field (2.28)–(2.30). The condition $V = 0$ gives $a_{0,\varphi}(\varphi) = 0$, that is

$$a_0(\varphi) = a_{00}. \tag{2.31}$$

Now it is convenient to restore the aforementioned Eq. (2.9); it gives the condition $b_{3,\varphi\varphi}(\varphi) + b_3(\varphi) + a_{31}z + c_1 = 0$ from which

$$a_{31} = 0, \tag{2.32}$$

$$b_3(\varphi) + c_1 = v_0 \cos(\varphi) - u_0 \sin(\varphi) \tag{2.33}$$

are deduced, u_0 and v_0 being integration constants.

Equalities (2.31) and (2.32) permit a dramatic simplification of relationship (2.30) as $w = h_2(\rho) - \rho a_{00} w_{\langle\rho z\rangle}(\rho, z)$; even better, embedding the first two addends in the last one, the final form

$$w(\rho, \varphi, z) = w_{\langle\rho z\rangle}(\rho, z) \tag{2.34}$$

is obtained, showing that w is *independent* of φ.

The updated displacement components are then:

$$u(\rho, \varphi, z) = u_{\langle\rho z\rangle}(\rho, z) + \{u_0 \cos(\varphi) + v_0 \sin(\varphi)\} + [a_{00}z + a_5(z) + c_2], \tag{2.35}$$

$$v(\rho, \varphi, z) = b_{00}\rho\varphi + \{e_1\rho - u_0 \sin(\varphi) + v_0 \cos(\varphi)\}, \tag{2.36}$$

$$w(\rho, \varphi, z) = w_{\langle\rho z\rangle}(\rho, z). \tag{2.37}$$

Note that, on one hand the terms enclosed in the square brackets in Eq. (2.35) can be embedded in $u_{\langle\rho z\rangle}(\rho, z)$; on the other hand, the terms enclosed in the brace brackets,

$$u_R := u_0 \cos(\varphi) + v_0 \sin(\varphi), \tag{2.38}$$

$$v_R := e_1\rho - u_0 \sin(\varphi) + v_0 \cos(\varphi), \tag{2.39}$$

are the components, in cylindrical coordinates, of a *rigid body motion*: a translation u_0 along the global X-axis, v_0 along the global Y-axis and a rotation e_1 about the Z-axis.

Therefore, the most general functional form of $\boldsymbol{u}$ in a curved beam under uniform bending is

$$u = u_{\langle\rho z\rangle}(\rho, z), \tag{2.40}$$

$$v = b_{00}\rho\varphi, \tag{2.41}$$

$$w = w_{\langle\rho z\rangle}(\rho, z), \tag{2.42}$$

modulo an unessential rigid body motion (cf. Mitchell, 1899).

2.7. Curved beam kinematics

The components of the local rigid rotation tensor, i.e. the skew-symmetric part of the displacement gradient, in cylindrical coordinates are (Reismann and Pawlik, 1980):

$$\omega_\rho := \frac{1}{2}\left(\frac{1}{\rho}w_{,\varphi} - v_{,z}\right) \equiv 0, \tag{2.43}$$

$$\omega_\varphi := \tfrac{1}{2}(u_{,z} - w_{,\rho}) = \tfrac{1}{2}(u_{\langle\rho z\rangle,z} - w_{\langle\rho z\rangle,\rho}), \tag{2.44}$$

$$\omega_z := \frac{1}{2}\left(\frac{1}{\rho}(\rho v)_{,\rho} - u_{,\varphi}\right) = b_{00}\varphi. \tag{2.45}$$

These relationships, together with the $\boldsymbol{u}$ expressions, permit to deduce several interesting qualitative considerations about the kinematical behavior of the curved beam subjected to bending moments:

1. *The cross-section remains plane*. Both the identity $\omega_\rho \equiv 0$ and Eq. (2.41) state that the cross-section does not warp.
2. *The cross-section undergoes local rigid rotations in its plane.* The rigid rotation ω_φ of a neighboring of every point of the cross-section is different from zero and it is a function of the point.
3. *The cross-section rotates about the Z-axis*. In fact it rotates by the angle $b_{00}\ \varphi$ about $z \equiv Z$, proportionally to the angular abscissa φ.
4. *Each longitudinal fiber remains circular, with its center on Z.* Since the displacement in the plane of the cross-section is independent of φ and the cross-section rotates rigidly about Z, then a longitudinal (circumferential) fiber remains circular and its center of curvature is always located on the Z-axis.
5. *Independence on the constitutive law, homogeneity and isotropy.* The results obtained on the kinematics of a curved beam under uniform bending are independent of the constitutive law (elastic, plastic, etc.); in particular the beam has to be neither homogeneous nor isotropic! The decisive *hypothesis* is that of *uniform bending*, i.e. the invariance of the geometric–mechanical characteristics with respect to φ; secondary, the constitutive equation must be compatible with the bending/shear–torsion uncoupling assumptions, $\gamma_{\rho\varphi} = \gamma_{\varphi z} \equiv 0$.

From a practical point of view, these remarks suggest how standard finite element codes can be exploited to simulate uniform bending in complex cross-section beams; that is, by considering a *small wedge* of the beam subject to a circumferential rigid rotation about Z of one face with respect to the other (leaving the in-plane cross-section displacements free).

3. Governing equilibrium equations

In this paragraph the field and boundary equilibrium equations will be derived in terms of the displacement components obtained above (for the sake of simplicity, functions $u_{\langle\rho z\rangle}$ and $w_{\langle\rho z\rangle}$ will be referred to in what follows by u and w, respectively, without ambiguity).

3.1. Stress components

The stress components, assuming that the beam is linearly elastic homogeneous and isotropic, are

$$\sigma_\rho = \frac{E}{(1+v)(1-2v)}\left[(1-v)u_{,\rho} + v\left(\frac{1}{\rho}u + w_{,z} + b_{00}\right)\right], \tag{3.1}$$

$$\sigma_\varphi = \frac{E}{(1+v)(1-2v)}\left[(1-v)\left(\frac{1}{\rho}u + b_{00}\right) + v(u_{,\rho} + w_{,z})\right], \tag{3.2}$$

$$\sigma_z = \frac{E}{(1+v)(1-2v)}\left[(1-v)w_{,z} + v\left(u_{,\rho} + \frac{1}{\rho}u + b_{00}\right)\right], \tag{3.3}$$

$$\tau_{\rho z} = \frac{E}{2(1+v)}(u_{,z} + w_{,\rho}). \tag{3.4}$$

3.2. Field equation in u and v

The second field equilibrium equation (*along* φ) is trivially fulfilled; the remaining ones,

$$\frac{1}{\rho}(\rho\sigma_\rho)_{,\rho} + \tau_{\rho z,z} - \frac{1}{\rho}\sigma_\varphi = 0, \tag{3.5}$$

$$\frac{1}{\rho}(\rho\tau_{\rho z})_{,\rho} + \sigma_{z,z} = 0, \tag{3.6}$$

after a bit of algebra, become

$$EQ_1 : 2(1-v)\left[\frac{1}{\rho}(\rho u)_{,\rho}\right]_{,\rho} + (1-2v)u_{,zz} + w_{,\rho z} = 2(1-2v)b_{00}\frac{1}{\rho}, \tag{3.7}$$

$$EQ_3 : (1-2v)\frac{1}{\rho}(\rho w_{,\rho})_{,\rho} + 2(1-v)w_{,zz} + \frac{1}{\rho}(\rho u)_{,\rho z} = 0. \tag{3.8}$$

3.3. Field equation in u

Let $EQ_1[u,w]$ and $EQ_3[u,w]$ denote symbolically Eqs. (3.5) and (3.6), respectively. Owing to the structure of the differential operator appearing in these relationships, it is not difficult to realize that the transformation

$$(EQ_3)_{,\rho z} - (1-2v)\left[\frac{1}{\rho}(\rho EQ_1)_{,\rho}\right]_{,\rho} - 2(1-v)(EQ_1)_{,zz} \tag{3.9}$$

eliminates the w function. The resulting PDE is more readable when the operator

$$\Omega := \partial_\rho\frac{1}{\rho}\partial_\rho\rho + \partial_{zz} \tag{3.10}$$

is introduced; so, the required equation is

$$\Omega\Omega u = 0. \tag{3.11}$$

3.4. Field equation in w

Similarly, the transformation

$$\frac{1}{\rho}(\rho EQ_1)_{,\rho z} - 2(1-v)\frac{1}{\rho}[\rho(EQ_3)_{,\rho}]_{,\rho} - (1-2v)(EQ_3)_{,zz} \tag{3.12}$$

eliminates the u function, yielding

$$\Lambda^*\Lambda^* w = 0. \tag{3.13}$$

The operator

$$\Lambda^* := \frac{1}{\rho}\partial_\rho\rho\,\partial_\rho + \partial_{zz} \tag{3.14}$$

is dual (in the sense defined in Section 6) of the operator $\Lambda := \rho\,\partial_\rho(1/\rho)\,\partial_\rho + \partial_{zz}$; it governs the bending problem in the stress approach, Eq. (5.7) in Part I and the fundamental field equation (4.7) derived in the next paragraph.

Obviously, the solutions of Eqs. (3.11) and (3.13) *are not independent*, because they must fulfill the original (lower order) equilibrium equations (3.5) and (3.6).

3.5. System of first-order differential equations

To classify the BVP, it can be useful to reduce Eqs. (3.7) and (3.8) to a system of first order PDEs (in this form, also the numerical solution of the problem is easier).

Using the trivial identity $2(1-\nu)-(1-2\nu)\equiv 1$, these equations can be rewritten as

$$2(1-\nu)\left[\frac{1}{\rho}(\rho u)_{,\rho}+w_{,z}\right]_{,\rho}+(1-2\nu)[u_{,z}-w_{,\rho}]_{,z}=2(1-2\nu)b_{00}\frac{1}{\rho}, \tag{3.15}$$

$$(1-2\nu)[\rho w_{,\rho}-\rho u_{,z}]_{,\rho}+2(1-\nu)[\rho w_{,z}+(\rho u)_{,\rho}]_{,z}=0. \tag{3.16}$$

By recognizing the similar terms enclosed in square brackets, the governing system of PDEs becomes

$$\frac{1}{\rho}(\rho u)_{,\rho}+w_{,z}=:\varepsilon'_V, \tag{3.17}$$

$$u_{,z}-w_{,\rho}=:\omega'_\varphi, \tag{3.18}$$

$$2(1-\nu)\varepsilon'_{V,\rho}+(1-2\nu)\frac{1}{\rho}(\rho\omega'_\varphi)_{,z}=2(1-2\nu)b_{00}\frac{1}{\rho}, \tag{3.19}$$

$$-(1-2\nu)\frac{1}{\rho}(\rho\omega'_\varphi)_{,\rho}+2(1-\nu)\varepsilon'_{V,z}=0, \tag{3.20}$$

in which the symbol ε'_V is used to emphasize its proximity to the volume strain ε_V, $\varepsilon'_V\equiv\varepsilon_V-b_{00}\,\varphi/\rho$; besides, $\omega'_\varphi\equiv 2\omega_\varphi$.

The following matrix form highlights the mathematical structure of the problem:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}\rho u\\w\end{bmatrix}_{,z}+\begin{bmatrix}0&-\rho\\1/\rho&0\end{bmatrix}\begin{bmatrix}\rho u\\w\end{bmatrix}_{,\rho}=\begin{bmatrix}\varepsilon'_V\\\rho\omega'_\varphi\end{bmatrix}, \tag{3.21}$$

$$\begin{bmatrix}2(1-\nu)&0\\0&(1-2\nu)/\rho\end{bmatrix}\begin{bmatrix}\varepsilon'_V\\\rho\omega'_\varphi\end{bmatrix}_{,z}+\begin{bmatrix}0&-(1-2\nu)/\rho\\2(1-\nu)&0\end{bmatrix}\begin{bmatrix}\varepsilon'_V\\\rho\omega'_\varphi\end{bmatrix}_{,\rho}=\begin{bmatrix}0\\2(1-2\nu)b_{00}/\rho\end{bmatrix}. \tag{3.22}$$

An elementary result must be now briefly recalled: the *characteristics* of a system of first order PDEs, in the form $\mathrm{A}\upsilon_{,\rho}+\mathrm{B}\upsilon_{,z}=0$, are defined as the curves that are solution of the ODE $\mathrm{d}z/\mathrm{d}\rho=k(\rho,z)$, where $k(\rho,z)$ is a root of the characteristic algebraic equation $det(\mathrm{A}+k\mathrm{B})=0$. In the case (3.21), this equation is unexpectedly independent of ρ, being $k^2+1=0$; for the second system (3.22), an *identical* algebraic equation is obtained.

Consequently, both systems are *uniformly elliptic* in the cross-section domain.

3.6. Formally uncoupled partial differential equation

Notwithstanding Eqs. (3.21) and (3.22) seem to be uncoupled, the BCs do not permit solving the problem in cascade by substituting the solution of the second system into the first one.

For the same reason it is not too effective to *reduce each system* to a pair of formally *uncoupled* PDEs; therefore, only the results are briefly summarized.

The auxiliary potential function ϕ is introduced to satisfy Eq. (3.20) as

$$2(1-\nu)\rho\varepsilon'_V = \phi_{,\rho}\,, \tag{3.23}$$

$$(1-2\nu)\rho\omega'_\varphi = \phi_{,z}\,, \tag{3.24}$$

while the previous equation (3.19) gives

$$\rho\left(\frac{1}{\rho}\phi_{,\rho}\right)_{,\rho} + \phi_{,zz} = 2(1-2\nu)b_{00}. \tag{3.25}$$

As an alternative, the system (3.22) can be reduced, through simple algebraic manipulations, to the following formally uncoupled (not independent) equations

$$\frac{1}{\rho}(\rho\varepsilon'_{V,\rho})_{,\rho} + \varepsilon'_{V,zz} = 0, \tag{3.26}$$

$$\left[\frac{1}{\rho}(\rho\omega'_\varphi)_{,\rho}\right]_{,\rho} + \omega'_{\varphi,zz} = 0, \tag{3.27}$$

that is, $\Lambda^*\varepsilon'_V = 0$ and $\Lambda^*\omega'_\varphi = 0$, respectively.

4. Fourth-order governing system in

In this paragraph the coupled Eqs. (3.7) and (3.8) in u and w are reduced to a single *fourth order* PDE; besides, a very interesting close connection with the stress formulation presented in Part I is achieved.

4.1. Displacement potential function Φ

The third equilibrium equation (3.8) is transformed into

$$[(1-2\nu)\rho w_{,\rho} + \rho u_{,z}]_{,\rho} + [2(1-\nu)\rho w_{,z}]_{,z} = 0, \tag{4.1}$$

to recognize a necessary existence condition of a potential function, $2(1-\nu)\Phi_{,zz}$ (a form that is suggested by the subsequent developments), such that

$$(1-2\nu)\rho w_{,\rho} + \rho u_{,z} = 2(1-\nu)(\Phi_{,zz})_{,z}\,, \tag{4.2}$$

$$2(1-\nu)\rho w_{,z} = -2(1-\nu)(\Phi_{,zz})_{,\rho}\,. \tag{4.3}$$

It is opportune to briefly mention that, despite the similarity with the equations defining the stress potential function ψ in Part I (cf. Eqs. (4.29) and (4.30) later), the *relationship* between Φ and ψ *is by no means immediate*, because the left hand side of these equations are not related to $\tau_{\rho z}$ and σ_z, respectively.

The latest equation is directly integrated with respect to z, giving

$$\rho w = -\Phi_{,\rho z}\,, \tag{4.4}$$

where the resulting arbitrary function of ρ is embedded in $\Phi_{,z\rho}$, without loss of generality. Substituting for w in the earlier equation, results in

$$-(1-2\nu)\left[\rho\left(\frac{1}{\rho}\Phi_{,\rho}\right)_{,\rho}\right]_{,z} + \rho u_{,z} = 2(1-\nu)\Phi_{,zzz}\,. \tag{4.5}$$

This equation can be quickly integrated, giving

$$\rho u = (1-2\nu)\rho\left(\frac{1}{\rho}\Phi_{,\rho}\right)_{,\rho} + 2(1-\nu)\Phi_{,zz} = (1-2\nu)\Lambda\Phi + \Phi_{,zz}\,, \tag{4.6}$$

in which the rising arbitrary function of ρ is included in Φ (without affecting its derivatives with respect to z, of course).

The remaining equilibrium equation is not yet satisfied. By substituting the stress components as a function of the displacement components u and w in Eq. (3.7), after a rather cumbersome algebra not reported here, the field equation is finally obtained in the following compact form

$$\Lambda\Lambda\Phi = b_{00}/(1-\nu), \quad \forall \mathbf{x} \in \mathscr{A}. \tag{4.7}$$

The field operator $\Lambda := \rho\,\partial_\rho(1/\rho)\partial_\rho + \partial_{zz}$ is that appearing in the *stress approach* presented in Part I.
In conclusion, owing to the identities (proofs are omitted)

$$\Lambda^*\Lambda^* w(\Phi) \equiv -\frac{1}{\rho}(\Lambda\Lambda\Phi)_{,\rho z}\,, \tag{4.8}$$

$$\Omega\Omega u(\Phi) \equiv (1-2\nu)\left[\frac{1}{\rho}(\Lambda\Lambda\Phi)_{,\rho}\right]_{,\rho} + 2(1-\nu)\frac{1}{\rho}(\Lambda\Lambda\Phi)_{,zz}\,, \tag{4.9}$$

it is straightforward to verify that the displacements (4.4) and (4.6) actually satisfy Eqs. (3.11), $\Omega\Omega u = 0$, and (3.13), $\Lambda^*\Lambda^* w = 0$, respectively.

4.2. Boundary conditions

The stress components (3.1)–(3.4) can be written in term of Φ, using the expression (4.6) and (4.4), as

$$\sigma_\rho \frac{1+\nu}{E} = \frac{\nu}{1-2\nu} b_{00} + (1-\nu)\left(\frac{1}{\rho}\Lambda\Phi\right)_{,\rho} + \left(\frac{1}{\rho}\Phi\right)_{,\rho zz} + \nu\frac{1}{\rho^2}\Lambda\Phi, \tag{4.10}$$

$$\sigma_z \frac{1+\nu}{E} = \frac{\nu}{1-2\nu} b_{00} + \nu\frac{1}{\rho}(\Lambda\Phi)_{,\rho} - \frac{1}{\rho}\Phi_{,\rho zz}\,, \tag{4.11}$$

$$\tau_{\rho z} \frac{1+\nu}{E} = (1-\nu)\frac{1}{\rho}(\Lambda\Phi)_{,z} - \left(\frac{1}{\rho}\Phi_{,\rho}\right)_{,\rho z} \equiv -\nu\frac{1}{\rho}(\Lambda\Phi)_{,z} + \frac{1}{\rho}\Phi_{,zzz}\,. \tag{4.12}$$

The static BCs ($\boldsymbol{n}$ being the outward normal to $\partial\mathscr{A}$) are

$$\sigma_\rho n_\rho + \tau_{\rho z} n_z = 0, \tag{4.13}$$

$$\tau_{\rho z} n_\rho + \sigma_z n_z = 0. \tag{4.14}$$

The second one, using $t_z \equiv n_\rho$ and $t_\rho \equiv -n_z$ ($\boldsymbol{t}$ being the tangent unit vector to $\partial\mathscr{A}$) and rearranging the result, writes as

$$\frac{1}{\rho}[-\nu(\Lambda\Phi)_{,z} + \Phi_{,zzz}]t_z + \frac{1}{\rho}[-\nu(\Lambda\Phi)_{,\rho} + \Phi_{,\rho zz}]t_\rho = \frac{\nu}{1-2\nu} b_{00}\, t_\rho. \tag{4.15}$$

This relationship, integrated along each closed curve c_i belonging to the boundary $\partial\mathscr{A}$, gives

$$\Phi_{,zz} - \nu\Lambda\Phi = \frac{\nu}{2(1-2\nu)} b_{00}\rho^2 + \kappa_i, \quad \forall \mathbf{x} \in c_i \subset \partial\mathscr{A}, \tag{4.16}$$

$\kappa_i, i = 1\ldots N$, being constants to be determined.

Using the first expression for $\tau_{\rho z}$, Eq. (4.13) can be written as

$$(1-\nu)\left(\frac{1}{\rho}\Lambda\Phi\right)_{,n} + \left(\frac{1}{\rho}\Phi_{,\rho z}\right)_{,t} + \left[\frac{\nu}{1-2\nu}b_{00} - \frac{1}{\rho^2}(\Phi_{,zz} - \nu\Lambda\Phi)\right]n_\rho = 0, \tag{4.17}$$

that, with Eq. (4.16), leads to the final form of the BC

$$(1-\nu)\left(\frac{1}{\rho}\Lambda\Phi\right)_{,n}+\left(\frac{1}{\rho}\Phi_{,\rho z}\right)_{,t}=\left(\frac{1}{\rho^2}\kappa_i-\frac{\nu}{2(1-2\nu)}b_{00}\right)t_z,\quad \forall \mathbf{x}\in c_i\subset\partial\mathscr{A}, \tag{4.18}$$

where $(\ )_{,t}$ denotes the derivative along the boundary.

Remarks. (1) Eqs. (4.16) and (4.18) constitute a pair of unstable BCs.

(2) Both are *degenerate* but the second one is in a simpler form; therefore, the structure of this BVP is not too different from that found in Part I.

(3) The *uniformity* conditions, $\int_{c_i} \mathrm{d}_t\Phi_{,zz}=0$ for each curve c_i constituting the boundary $\partial\mathscr{A}$, are necessarily fulfilled by the implicit assumption that Φ is regular enough, together with its required derivatives.

(4) The special solution $\Phi_{\mathrm{R}} := a\rho^2 z$ is a *rigid body translation* along Z, not affecting the constants κ_i.

(5) Note that the substitution $\Phi \leftarrow \Phi + \kappa_1\,(z^2/2(1-\nu))$ cancels κ_1 in Eq. (4.16); nonetheless it affects the u displacement (4.6) and Eq. (4.18). Consequently, $\kappa_1 \equiv 0$ cannot be assumed, in general.

4.3. Discrete compatibility conditions

In conclusion, the conditions determining the unknown constants κ_i are discussed. By integrating the relationship (4.18) along the closed curve $c_i \subset \partial\mathscr{A}$, results in

$$(1-\nu)\oint_{c_i}\left(\frac{1}{\rho}\Lambda\Phi\right)_{,n}\mathrm{d}l+\oint_{c_i}\mathrm{d}_t\left(\frac{1}{\rho}\Phi_{,\rho z}\right)=\oint_{c_i}\left(\kappa_i\frac{1}{\rho^2}-\frac{\nu}{2(1-2\nu)}b_{00}\right)t_z\,\mathrm{d}l. \tag{4.19}$$

Now, consider that:

(i) Eq. (4.4) states that $(1/\rho)\,\Phi_{,\rho z}\equiv -w$; then the second integral is nil by the *continuity* of the longitudinal displacement;
(ii) since $t_z\,\mathrm{d}l = \mathrm{d}_t z$, then $\oint_{c_i} t_z\,\mathrm{d}l \equiv 0$; thus the constant b_{00} disappears in the relationships (4.19);
(iii) $\alpha_{c_i} := \oint_{c_i}(1/\rho^2)\,t_z\,\mathrm{d}l$; by denoting with $\mathrm{Int}(c_i)$ the portion of the surface of the plane delimited by the closed curve c_i (cf. Fig. 4 in Part I), these geometrical parameters can be also computed as follows:

$$\alpha_{c_1}=\oint_{c_1\subset\partial\mathscr{A}}\frac{1}{\rho^2}n_\rho\,\mathrm{d}l=+\oint_{c_1\equiv\partial\mathrm{Int}(c_1)}\frac{1}{\rho^2}n_\rho\,\mathrm{d}l=-2\int_{\mathrm{Int}(c_1)}\frac{1}{\rho^3}\,\mathrm{d}a<0, \tag{4.20}$$

$$\alpha_{c_i}=\oint_{c_i\subset\partial\mathscr{A}}\frac{1}{\rho^2}n_\rho\,\mathrm{d}l=-\oint_{c_i\equiv\partial\mathrm{Int}(c_i)}\frac{1}{\rho^2}n_\rho\,\mathrm{d}l=+2\int_{\mathrm{Int}(c_i)}\frac{1}{\rho^3}\,\mathrm{d}a>0, \tag{4.21}$$

for the outermost boundary c_1 and the inner curves c_i, respectively ($\boldsymbol{n}$ is the outward normal to the considered domain).

By collecting these results, the anticipated conditions on the unknown constants κ_i can be written as

$$(1-\nu)\oint_{c_i}\left(\frac{1}{\rho}\Lambda\Phi\right)_{,n}\mathrm{d}l=\alpha_{c_i}\kappa_i,\quad \forall\ c_i\subset\partial\mathscr{A},\quad i=1\ldots N. \tag{4.22}$$

that, owing to the first remark, assumes the meaning of discrete compatibility conditions.

In order to show that these equations are not independent, it is useful to rewrite them in the following equivalent form

$$(1-\nu)\oint_{c_i}\frac{1}{\rho}(\Lambda\Phi)_{,n}\,\mathrm{d}l+\frac{1-2\nu}{\nu}\alpha_{c_i}\kappa_i=0. \tag{4.23}$$

In fact, consider that $v(1/\rho)(\Lambda\Phi)_{,n} \equiv n_\rho(-v(1/\rho^2)\Lambda\Phi) + v(1/\rho)(\Lambda\Phi)_{,n}$. Substituting Eq. (4.16) in the first term at the right hand side and integrating along c_i results in

$$v\oint_{c_i}\left(\frac{1}{\rho}\Lambda\Phi\right)_{,n} \mathrm{d}l = -\oint_{c_i}\left(\frac{1}{\rho^2}\Lambda\Phi_{,z}\right)_{,z} t_z \,\mathrm{d}l + \frac{v}{2(1-2v)} b_{00}\oint_{c_i} n_\rho \,\mathrm{d}l + \kappa_i \oint_{c_i}\frac{1}{\rho^2} n_\rho \,\mathrm{d}l + v\oint_{c_i}\frac{1}{\rho} \times (\Lambda\Phi)_{,n}\,\mathrm{d}l = \kappa_i\alpha_{c_i} + v\oint_{c_i}\frac{1}{\rho}(\Lambda\Phi)_{,n}\,\mathrm{d}l, \tag{4.24}$$

since the first two integral are zero. Multiplying this result by $(1-v)/v$ and subtracting Eq. (4.22), the relationship (4.23) is proved.

Theorem. *The discrete boundary conditions are non independent, since constants κ_i satisfy the equation*

$$(1-2v)\sum_{i=1}^{N}\kappa_i\alpha_{c_i} + vb_{00}\oint_{c_i}\frac{1}{\rho}\,\mathrm{d}a = 0. \tag{4.25}$$

Proof. The field equation (4.7), divided by ρ and integrated in the $\mathscr{A}$ domain, gives

$$\int_{\mathscr{A}}\left\{\left[\frac{1}{\rho}(\Lambda\Phi)_{,\rho}\right]_{,\rho} + \left(\frac{1}{\rho}\Lambda\Phi_{,z}\right)_{,z}\right\}\mathrm{d}a = \frac{b_{00}}{1-v}\int_{\mathscr{A}}\frac{1}{\rho}\,\mathrm{d}a, \tag{4.26}$$

or

$$(1-v)\int_{\partial\mathscr{A}}\frac{1}{\rho}(\Lambda\Phi)_{,n}\,\mathrm{d}l - b_{00}\int_{\mathscr{A}}\frac{1}{\rho}\,\mathrm{d}a = 0. \tag{4.27}$$

On the other hand, the sum of the N relationship (4.23) is

$$(1-v)\int_{\partial\mathscr{A}}\frac{1}{\rho}(\Lambda\Phi)_{,n}\,\mathrm{d}l + \frac{1-2v}{v}\sum_{i=1}^{N}\alpha_{c_i}\kappa_i = 0. \tag{4.28}$$

By comparing the last two results, Eq. (4.25) is obtained. □

The comparison with the torsion problem for straight beams, the shear–torsion problem (Mentrasti, 1996) or the stress formulation of the problem in question (*discrete compatibility conditions* in Section 5 of Part I) for a curved beam is expressive: in those cases one condition is simply a linear combination of the remaining ones (cf. also Remark 5 on κ_1, above).

4.4. Link to the stress potential function ψ

A connection of the present formulation with the study presented in Part I is possible by interpreting the third equilibrium equation (3.6) as a condition for the existence of a stress potential such that

$$\sigma_z = -\frac{1}{\rho}\psi_{,\rho}\,, \tag{4.29}$$

$$\tau_{\rho z} = \frac{1}{\rho}\psi_{,z}\,. \tag{4.30}$$

Furthermore, *it has been identified* with the *stress potential function* ψ defined in Section 2 of Part I and for this reason it is assumed to be known in the following.

Consequently, from Eqs. (3.3) and (3.4)

$$\sigma_z\colon \frac{E}{(1+v)(1-2v)}\left[(1-v)w_{,z} + v\frac{1}{\rho}(\rho u)_{,\rho} + vb_{00}\right] = -\frac{1}{\rho}\psi_{,\rho}\,, \tag{4.31}$$

$$\tau_{\rho z}\colon \frac{E}{2(1+v)}(w_{,\rho} + u_{,z}) = \frac{1}{\rho}\psi_{,z} \tag{4.32}$$

hold true (see also Eqs. (2:3), (2:7) and (2:8) of Part I).

From these relationships, it is easy to derive two uncoupled PDEs in u and w, by eliminating the common mixed derivatives. However, since the equations so obtained cannot be independent, of course, a more efficient strategy is to rewrite the second equation as

$$w_{,\rho} + \left(u - 2\frac{(1+v)}{E}\frac{1}{\rho}\psi\right)_{,z} = 0, \tag{4.33}$$

in such a way that it can be regarded as a necessary condition for the existence of a new potential function F, solving Eq. (4.33) as

$$u = -F_{,\rho} + 2\frac{(1+v)}{E}\frac{1}{\rho}\psi, \tag{4.34}$$

$$w = F_{,z}\,. \tag{4.35}$$

Substituting these expressions in the remaining Eq. (4.31), the field equation in F is obtained

$$v\frac{1}{\rho}(\rho F_{,\rho})_{,\rho} - (1-v)F_{,zz} = \frac{(1+v)}{E}\frac{1}{\rho}\psi_{,\rho} + v\,b_{00}. \tag{4.36}$$

It is very interesting to note that:

1. Eq. (4.36) is *hyperbolic*: its characteristics are the straight lines $z = \pm k\rho$, with $k^2 := (1-v)/v$. The occurrence of such a class of PDEs is very unusual in dealing with *static* linear elastic problems!
2. There are *no boundary or initial conditions* associated to this equation: in fact all the *equilibrium* BCs are already fulfilled by the ψ function by hypothesis; on the other hand, no additional physical requirements can be imposed.
3. The only meaningful solution for this hyperbolic equation is a *particular* solution: in Appendix B a simple procedure is presented to obtain such a result, irrespective of the cross section geometry.

The alternative expressions (4.11), (4.12) and (4.29), (4.30) for the stress σ_z and $\tau_{\rho z}$ permit determining the relationship between the functions ψ and Φ as follows:

$$\frac{(1+v)}{E}\psi \equiv -v\Lambda\Phi + \Phi_{,zz} - \frac{v}{2(1-2v)}b_{00}\rho^2 - \kappa_*, \tag{4.37}$$

where κ_* is a constant. Furthermore,

$$\kappa_i - \kappa_* = \frac{(1+v)}{E}\psi_i^0, \quad i = 1\ldots N. \tag{4.38}$$

can be drawn from the BCs (4.16).

When $\psi_1^0 \equiv 0$ is assumed, then κ_* can be identified as κ_1.

5. Force resultants

Since the bending problem for the curved beam is solved exactly, the computation of force resultants N_φ, V_ρ, V_z and M_η is not essentially different from what was presented in Part I using the stress method: here it is enough to directly consider $\tau_{\rho z}$ *without resorting to the potential stress function* ψ (together, of course, with the relevant BC).

For M_ρ the same method is not available so that it can be useful to derive it by invoking only the equilibrium condition (i.e. without making use of any explicit solution).

5.1. Moment component along the ρ-axis

With the expression of σ_φ derived from the equilibrium equation (3.5), the moment M_ρ can be also written as

$$\begin{aligned} M_\rho &:= \int_{\mathscr{A}} Z\sigma_\varphi \,\mathrm{d}a = \int_{\mathscr{A}} Z[(\rho\sigma_\rho)_{,\rho} + \rho\tau_{\rho z,z}]\,\mathrm{d}a = \int_{\mathscr{A}} [(\rho Z\sigma_\rho)_{,\rho} + (\rho Z\tau_{\rho z})_{,z} - \rho\tau_{\rho z}]\,\mathrm{d}a \\ &= \int_{\partial\mathscr{A}} Z\rho[\sigma_\rho n_\rho + \tau_{\rho z} n_z]\,\mathrm{d}l - \int_{\mathscr{A}} \rho\tau_{\rho z}\,\mathrm{d}a = -\int_{\mathscr{A}} \rho\tau_{\rho z}\,\mathrm{d}a = 0, \end{aligned} \tag{5.1}$$

where the BC (4.13) is used to cancel the integral along the boundary, while the last equality is proved in what follows.

Consider the BC (4.14) multiplied by ρ^2 and integrated on $\partial\mathscr{A}$:

$$0 \equiv \int_{\partial\mathscr{A}} [\rho^2\tau_{\rho z} n_\rho + \rho^2\sigma_z n_z]\,\mathrm{d}l = \int_{\mathscr{A}} [(\rho^2\tau_{\rho z})_{,\rho} + (\rho^2\sigma_z)_{,z}]\,\mathrm{d}a, \tag{5.2}$$

from the equilibrium equation (3.6), $\sigma_{z,z} = -(1/\rho)(\rho\tau_{\rho z})_{,\rho}$ results, so that the above relationship finally gives

$$\int_{\partial\mathscr{A}} \left[(\rho^2\tau_{\rho z})_{,\rho} - \rho^2\frac{1}{\rho}(\rho\tau_{\rho z})_{,\rho}\right]\mathrm{d}a = \int_{\partial\mathscr{A}} \rho\tau_{\rho z}\,\mathrm{d}a = 0. \tag{5.3}$$

6. Attempt for a variational formulation

The advantage of a variational formulation for a linear differential problem lies first in a relative easy proof of the existence of the solution and, secondly, in the possibility of an (almost standard) finite element approach to searching for the solution itself.

In this section an attempt is made to solve the so-called *inverse problem* of the calculus of variation, i.e. finding $f[H]$ and $g[H]$ such that, from stationary conditions for the functional

$$\int_{\mathscr{A}} f(H, H_{,i}, H_{,ij})\,\mathrm{d}a + \int_{\partial\mathscr{A}} g(H, H_{,t}, H_{,tt})\,\mathrm{d}l, \tag{6.1}$$

both the *field equation* and *the boundary conditions* of the problem can be derived.

Even though a positive answer to this question cannot be found, it is very interesting to present the results attained because they enlighten the mathematical structure of the differential system governing the bending curved beams.

6.1. Functional associated to the field equation

Reference will be made to a field equation of the form

$$\Lambda\Lambda v = c \tag{6.2}$$

appearing here and in Part I.

Since the higher order derivatives involved in $\Lambda\Lambda$ are $[\rho((1/\rho)H_{,\rho})_{,\rho}]_{,\rho\rho}$ and $H_{,zzzz}$ a functional giving such terms can be assumed to be proportional to a (non-homogeneous) quadratic form of $\rho((1/\rho)H_{,\rho})_{,\rho}$ and $H_{,zz}$, modulo a multiplicative function of ρ:

$$f[H] := \frac{1}{2}\lambda(\rho)\left\{\left(\left[\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho}\right]_{,\rho\rho}\right)^2 + (H_{,zz})^2 + 2\mu\rho\left(\frac{1}{\rho}H_{,\rho}\right)_{,\rho}H_{,zz}\right\} - \chi(\rho)c_0H, \tag{6.3}$$

where $\lambda(\rho)$, $\chi(\rho)$ and the constant μ have to be determined.

Leaving aside a detailed proof, involving only elementary mathematics, the result obtained is:

Lemma 1. *When $\lambda(\rho) = \chi(\rho) := 1/\rho$ and $\mu := 1$ the Euler equation*

$$f_{,H} - (f_{,H_{,\rho}})_{,\rho} - (f_{,H_{,z}})_{,z} + (f_{,H_{,\rho\rho}})_{,\rho\rho} + (f_{,H_{,\rho z}})_{,\rho z} + (f_{,H_{,zz}})_{,zz} = 0, \tag{6.4}$$

coincides with the required field equation, divided by ρ.

6.2. Variational conditions at the boundary

The first BC is

$$\left\{[f_{,H_{,\rho}} - (f_{,H_{,\rho\rho}})_{,\rho} - (f_{,H_{,\rho z}})_{,z}]n_\rho + [f_{,H_{,z}} - (f_{,H_{,z}})_{,z} - (f_{,H_{,\rho z}})_{,\rho}]n_z - (t_i f_{,H_{,\xi_i\xi_j}} n_j)_{,t}\right\}\delta H = 0, \tag{6.5}$$

in which, for the sake of brevity, the tangent unit vector t at the boundary is supposed to be continuous (sum on repeated indexes $i, j = 1 \dots 2$ in the last addend).

The second can be written as

$$(n_i f_{,H_{,\xi_i\xi_j}} n_j)\delta H_{,n} = 0, \tag{6.6}$$

where $H_{,n}$ is the derivative of H along the normal.

It is not difficult to understand that none of the *unstable and degenerate* BCs involved in the *bending* problem can be by no means derived from the chosen f (or any g).

The conclusion is: the *bending* problem (irrespective of the stress or bending approach) is a *non-variational* problem, due to the occurrence of *degenerate unstable* BCs.

6.3. Adjoint operators

Some simple integral identities will be derived in this section to show the relationships between the differential operators appearing in the different formulations presented above.

Lemma 2. *Let u and v be two sufficiently regular functions defined in $\mathscr{A}$, with regular boundary. Then the following identities hold true*:

$$\int_{\mathscr{A}} u\Lambda v\,\mathrm{d}a = \int_{\mathscr{A}} v\Lambda^* u\,\mathrm{d}a + \int_{\partial\mathscr{A}}\left[u v_{,n} - v\frac{1}{\rho}(\rho u)_{,n}\right]\mathrm{d}l, \tag{6.7}$$

$$\int_{\mathscr{A}} u\frac{1}{\rho}\Lambda v\,\mathrm{d}a = \int_{\mathscr{A}} v\frac{1}{\rho}\Lambda u\,\mathrm{d}a + \int_{\partial\mathscr{A}} \frac{1}{\rho}[u\,v_{,n} - v u_{,n}]\,\mathrm{d}l, \tag{6.8}$$

$$\int_{\mathscr{A}} u\,\Omega v\,\mathrm{d}a = \int_{\mathscr{A}} v\Lambda u\,\mathrm{d}a + \int_{\partial\mathscr{A}} \left[u\frac{1}{\rho}(\rho v)_{,n} - v\,u_{,n}\right]\mathrm{d}l. \tag{6.9}$$

Thus, the algebraic adjoint of the operator $\Lambda := \rho\partial_\rho(1/\rho)\partial_\rho + \partial_{zz}$ is both $\Lambda^* := (1/\rho)\partial_\rho\rho\partial_\rho + \partial_{zz}$ (see Eq. (3.13)) and $\Omega := \partial_\rho(1/\rho)\partial_\rho\rho + \partial_{zz}$ (see Eq. (3.11)). Vice versa $(1/\rho\Lambda)$ is formally *self-adjoint*.

Finally, comparing the last two identities results in

$$\int_{\mathscr{A}} u\,\Omega v\,\mathrm{d}a = \int_{\mathscr{A}} v\Lambda^* u\,\mathrm{d}a, \tag{6.10}$$

independently of the boundary conditions.

7. Conclusion

The circular curved beam, under *uniform bending* in its plane, is investigated by a *displacement method*. The 3D problem is solved exactly: the cross-section remains plane and rotates about the Z-axis, while each longitudinal fiber remains circular.

The obtained field displacement, (2.40)–(2.42), being *independent* of any assumption regarding the *material*, permits extending the subsequent analysis to *anisotropic composite* curved beams whose mechanical properties are *invariant* with respect to the *symmetry group* $\mathscr{G}_Z$ of the rotations about the Z-axis.

The equilibrium equation (in several form) and the BCs for the homogeneous linear elastic body are derived: they constitute an *unstable degenerate* BVP with a structure similar to that found in Part I.

The major difficulty, namely the *degenerate character* of the BVP, is an inherent quality of the problem, in both displacement and stress formulations.

The *non-variational* nature of this kind of problem is also detected.

7.1. Future developments

Bending of curved beams, even in the simple homogeneous and elastic case, turns out to be an *unexpectedly formidable problem*: a variable coefficient, fourth order, elliptic, partial differential field equation with two unstable degenerate BCs.

Since the physical problem seems to be regular (also with regard to the very simple behavior of the straight beams under bending) several inquiries could be carried out:

- search for alternative formulations leading to *well posed* BVPs (integral transform, boundary integro-differential equation);
- determine the conditions for the existence of the solution; because this kind of BVP is not Fredholm type, precise the general form of the BCs so that they are compatible;
- detect in which cases the discontinuity of the normal at the boundary induces *stress concentration* (in homogeneous or composite materials);
- carry out a special theory for *thin-walled*, highly connected, beams.

Appendix A. Functional form of the {\it v} component

The basic tools for deriving an expression for $v(\rho, \varphi, z)$ is the following elementary *lemma*: the general solution of a PDE of the form

$$v(\lambda, \mu)_{,\lambda\mu\mu} = 0 \tag{A.1}$$

is

$$v(\lambda, \mu) = \mu v^1_{\langle\lambda\rangle}(\lambda) + v^2_{\langle\lambda\rangle}(\lambda) + v_{\langle\mu\rangle}(\mu), \tag{A.2}$$

in which, for the sake of conciseness, indexes enclosed in angle brackets $\langle\ldots\rangle$ are used both to denominate the function and to explicitly show the independent variables.

Then, Eq. (2.6), $v_{,\varphi zz} = 0$, gives

$$v = v_{\langle\rho\varphi\rangle}(\rho, \varphi) + v_{\langle\varphi z\rangle}(\varphi, \mathscr{Z}) + v_{\langle\rho z\rangle}(\rho, \mathscr{Z}). \tag{A.3}$$

Substituting this expression for v in Eqs. (2.1), (2.3), (2.10), the following relationships must hold

$$v_{\langle\varphi z\rangle}{}_{,\varphi zz} = 0, \tag{A.4}$$

$$v_{\langle\rho\varphi\rangle}{}_{,\rho\rho\varphi} = 0, \tag{A.5}$$

$$v_{\langle\rho\varphi\rangle}{}_{,\rho\varphi\varphi} = 0. \tag{A.6}$$

The first equation gives

$$v_{\langle\varphi z\rangle} = \mathscr{Z} a_{0,\varphi}(\varphi) + a_1(\varphi) + a_2(\mathscr{Z}), \tag{A.7}$$

the second one gives

$$v_{\langle\rho\varphi\rangle} = \rho b_0(\varphi) + b_1(\varphi) + b_2(\rho). \tag{A.8}$$

The third condition specifies that $b_{0,\varphi\varphi}(\varphi) = 0$, from which $b_0(\varphi) := b_{00}\,\varphi + b_{01}$, so that $v_{\langle\rho\varphi\rangle}$ can be written

$$v_{\langle\rho\varphi\rangle} = b_{00}\rho\varphi + b_1(\varphi) + b_2(\varphi), \tag{A.9}$$

the $b_{01}\ \rho$ term being enclosed in $b_2(\rho) \leftarrow b_2(\rho) + b_{01}\rho$.

The final form of the v displacement is

$$v(\rho, \varphi, \mathscr{Z}) = b_{00}\,\rho\varphi + b_2(\rho) + b_3(\varphi) + \mathscr{Z}, a_{0,\varphi}(\varphi) + a_2(\mathscr{Z}) + v_{\langle\rho z\rangle}(\rho, \mathscr{Z}), \tag{A.10}$$

in which $b_3(\varphi) := a_1(\varphi) + b_1(\varphi)$ and the (arbitrary) function $a_0(\varphi)$ appears under the derivative sign to simplify the form of the ensuing Eq. (2.24).

Appendix B. Solution of the hyperbolic partial differential equation (4.36)

The displacement components u and w are obtained, in Eqs. (4.34) and (4.35), as simple functions of both the potential stress function ψ and the new function F; ρF must be a solution of Eq. (4.36) or, equivalently,

$$\frac{v}{(1+v)}(\rho F_{,\rho})_{,\rho} - \rho F_{,zz} = \psi. \tag{B.1}$$

Integration along the characteristics is not more effective than a direct attack of the equation by a numerical method.

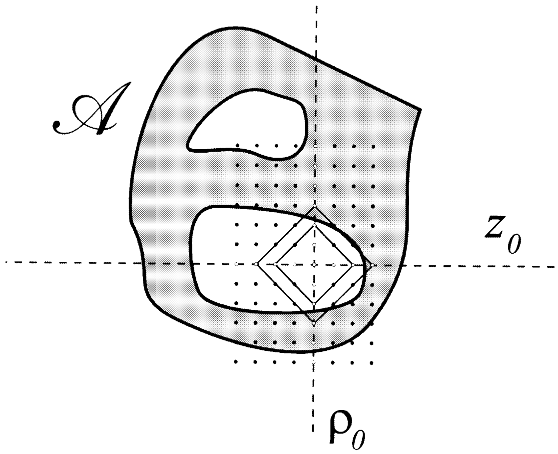

Fig. A1. Finite difference scheme for Eq. (4.36).

Since no initial or boundary conditions must be considered, a finite difference approach is very simple to implement, irrespective of the geometry of the multi-connected cross-section:

1. Let ψ be prolonged with continuity through $\partial\mathscr{A}$ by defining $\psi = \psi_i^0$ within each simply connected region of $\mathscr{E}^2/\mathscr{A}$ bordered by the curve $c_i \subset \partial\mathscr{A}$ (ψ_1^0 on the outer region up to infinity).
2. Arbitrary (continuous) values $F(\rho, \mathscr{Z}_0)$ and $F(\rho_0, \mathscr{Z}_0)$ are assigned on two *orthogonal* lines $\mathscr{Z} = z_0$ and $\rho = \rho_0$, respectively, with the only constraint that Eq. (4.36) be satisfied at the point $(\rho_0, \mathscr{Z}_0)$; e.g. $F(\rho, \mathscr{Z}_0) \equiv 0$ and

$$-(1-v)F(\rho_0, \mathscr{Z}) = \frac{(1+v)}{E}\int_{\mathscr{Z}_0}^{\mathscr{Z}} (\mathscr{Z} - \zeta)\frac{1}{\rho_0}\psi,_{\rho}(\rho_0, \zeta)\,\mathrm{d}\zeta + \frac{1}{2}v b_{00}(\mathscr{Z} - \mathscr{Z}_0)^2. \tag{B.2}$$

3. Finally, F is computed on a $\{\delta\rho, \delta\mathscr{Z}\}$ grid by writing the 3×3 discrete finite difference operator, associated to Eq. (B.1), at each point located near $(\rho_0, \mathscr{Z}_0)$ and then by proceeding, layer by layer (see. Fig. 1), until all points belonging to the cross-section are exhausted.

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